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AN EXAMPLE OF COMPACT EXTENSIONS OF KRONECKER FACTORS AND SKEW PRODUCTS OF IRRATIONAL ROTATIONS WITH FINITE GROUPS

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ABSTRACT

We give an example to show that compact extensions of Kronecker factors for two ergodic commuting measure preserving transformations can be different. Also, a criteria for ergodicity of skew products of irrational rotations with finite abelian groups is obtained.

1. Introduction

Compact extension for a dynamical system $(X, \mathcal{B}, \mu, S)$ with respect to a factor has been much studied. In this note, we will focus on a property for compact extension with respect to the Kronecker factor, which will be called the degree 2 Kronecker factor and denoted by $\mathcal{K}_2(S)$ (see §2 for more details).

Let S and T be ergodic commuting measure preserving transformations on a probability space $(X, \mathcal{B}, \mu)$. It is well known that $\mathcal{K}_1(S)$, the Kronecker factor of S, is the same as $\mathcal{K}_1(T)$, the Kronecker factor of T. But, generally speaking, $\mathcal{K}_2(S) \neq \mathcal{K}_2(T)$. In §4, we will give an example. Our proof of this example depends on the ergodicity of a certain type of skew products of irrational rotations with finite abelian groups, and problems in this area have been studied by Oren [6], Schmidt [8] and Veech [9]. In §3, we will also give a criterion for ergodicity. Most of the material in §2 can be found in [1, 2, 11] and, for the readers' convenience, we give a brief summary.

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2. Compact extensions and cocycles

We start from some facts about factors and compact extensions. Readers can find more details in [1, 2, 11].

Let $(X, \mathcal{B}, \mu)$ be a probability space with a measure preserving transformation S and let $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$. A factor of $(X, \mathcal{B}, \mu, S)$ is a S-invariant sub-σ-algebra $\mathcal{C} \subset \mathcal{B}$. Let

$$\mathcal{H}_1(S) = \{f \in \mathcal{H}\colon \text{ for any } \varepsilon > 0, \exists\, c_1, \ldots, c_m \in L^\infty(X, \mathcal{B}, \mu) \text{ s.t. } \min_{1 \le i \le m} |\, S^n f - c_i| < \varepsilon \text{ for all } n\}.$$

Then there is a S-invariant sub-σ-algebra $\mathcal{K}_1(S) \subset \mathcal{B}$ such that

$$\mathcal{H}_1(S) = L^2(X, \mathcal{K}_1(S), \mu).$$

$\mathcal{K}_1(S)$ is called the **degree 1 Kronecker factor** (or just **Kronecker factor**) of the dynamical system $(X, \mathcal{B}, \mu, S)$. Actually, there exists a compact abelian group with a rotation equivalent (in some literature, called conjugate, see [10, page 59]) to this Kronecker factor. It is well known that if S and T are ergodic commuting (i.e. $ST = TS$) measure preserving transformations on X, then $\mathcal{K}_1(S) = \mathcal{K}_1(T)$.

Let $\mathcal{C}$ be a factor of $(X, \mathcal{B}, \mu, S)$. Define

$$\mathcal{H}_{\mathcal{C}}(S) = \{f \in \mathcal{H}\colon \text{ for any } \varepsilon > 0, \exists\, c_1, \ldots, c_m \in L^\infty(X, \mathcal{B}, \mu) \text{ s.t. } \min_{1 \le i \le m} \mathbf{E}((S^n f - c_i)^2 \mid \mathcal{C})(x) < \varepsilon \text{ for a.e. } x \in X \text{ and all } n\}.$$

Then there is a S-invariant sub-σ-algebra $\mathcal{C}_S \subset \mathcal{B}$ such that

$$\mathcal{H}_{\mathcal{C}}(S) = L^2(X, \mathcal{C}_s, \mu).$$

$\mathcal{C}_S$ will be called a compact extension of $\mathcal{C}$. In particular, the compact extension of the Kronecker factor of S will be called the **degree 2 Kronecker factor** and denoted by $\mathcal{K}_2(S)$. It is clear that $\mathcal{K}_2(S) \supset \mathcal{K}_1(S)$. Unlike the degree 1 Kronecker factors, for two ergodic commuting measure preserving transformations S and T, $\mathcal{K}_2(S)$ is usually not the same as $\mathcal{K}_2(T)$. We will give an example in §4. The following theorem can be used to test if a function is in the compact extension. A proof can be found in [2, pages 62–64].

THEOREM 2.1: *Let $\mathcal{C}$ be a factor of ergodic system $(X, \mathcal{B}, \mu, T)$. Then $f \in \mathcal{H}$ is perpendicular to $\mathcal{H}_{\mathcal{C}}(S)$ if and only if, for any $g \in L^\infty(X, \mathcal{B}, \mu)$,*

$$\frac{1}{N} \sum_{n=0}^{N-1} |\mathbf{E}(gS^n f \mid \mathcal{C})(x)| \to 0$$

in the sense of almost pointwise and L^1.

Now we consider two examples which represent two extreme situations for compact extensions of Kronecker factors.

Example 1: Let $X = \mathbf{T} \times \mathbf{Z}_2^\infty$. For measurable function $\mathbf{F}\colon \mathbf{T} \to \mathbf{Z}_2^\infty$ and an irrational number $\alpha \in \mathbf{T}$, one can define a measure preserving transformation S on X by

$$S(t, \mathbf{x}) = (t + \alpha, \mathbf{x} + \mathbf{F}(t)).$$

Now we assume that S is ergodic. Then

$$\mathcal{K}_1(S) = \{A \times \mathbf{Z}_2^\infty\colon\ A \text{ is a measurable subset of } \mathbf{T}\}.$$

Let $\gamma_j(\mathbf{x}) = (1 - 2x_j)$ and

$$\gamma_{j_1 j_2 \cdots j_m} = \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_m}$$

for all $j_1, j_2, \ldots, j_m$. These are all characters for $\mathbf{Z}_2^\infty$. Let $c_1 = \gamma_{j_1 j_2 \cdots j_m}$ and let $c_2 = -\gamma_{j_1 j_2 \cdots j_m}$. For any $(t, \mathbf{x}) \in X$, $\gamma_{j_1 j_2 \cdots j_m}(\mathbf{F}(t))$ is either 1 or -1. Noticing that

$$S^n \gamma_{j_1 j_2 \cdots j_m}(t, \mathbf{x}) = \gamma_{j_1 j_2 \cdots j_m}(\mathbf{F}(t)) \gamma_{j_1 j_2 \cdots j_m}(\mathbf{x}),$$

we have that

$$\min_{1,2} \mathbf{E}((S^n \gamma_{j_1 j_2 \cdots j_m} - c_i)^2 \mid \mathcal{K}_1(S)) = 0.$$

Since linear combinations of characters are dense in $L^2(X, \mathcal{B}, \mu)$, $\mathcal{K}_2(S) = \mathcal{B}$. ∎

Example 2: Let $X = \mathbf{T} \times \mathbf{Z}_2^\infty$. For any $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$, define a shift $\tau: \mathbf{Z}_2^\infty \to \mathbf{Z}_2^\infty$ by $(\tau\mathbf{x})_i = x_{i+1}$. Then for any irrational number $\beta \in \mathbf{T}$, one can define a measure preserving transformation T on X by

$$T(t, \mathbf{x}) = (t + \beta, \tau\mathbf{x}).$$

Since τ is mixing on $\mathbf{Z}_2^\infty$, T must be ergodic. One can easily find out that

$$\mathcal{K}_1(T) = \{A \times \mathbf{Z}_2^\infty \colon\ A \text{ is a measurable subset of } \mathbf{T}\}.$$

For any $g \in L^\infty(X, \mathcal{B}, \mu)$,

$$\frac{1}{N}\sum_{n=0}^{N-1} |\mathbf{E}(gS^n\gamma_{j_1j_2\cdots j_m} \mid \mathcal{C})(t)| = \frac{1}{N}\sum_{n=0}^{N-1} \left|\int (g(t,\mathbf{x})\gamma_{j_1j_2\cdots j_m}(\tau^n\mathbf{x})d\mathbf{x}\right| \to 0$$

for all $t \in \mathbf{T}$. Noticing that g and $\gamma_{j_1j_2\cdots j_m}$ are all bounded functions, by Theorem 2.1, we have that $\mathcal{K}_2(T) = \mathcal{K}_1(T)$. ∎

Let $\rho_\alpha: t \to t + \alpha$ be a rotation on $\mathbf{T}$ defined by an irrational number α. For any function f from $\mathbf{T}$ to a compact abelian group $\mathbf{G}$, one can define a cocycle by

$$f^{(n)}(t) = \sum_{k=0}^{n-1} f(t + k\alpha).$$

For brevity, we will sometimes call f a cocycle. For any two cocycles f_1 and f_2, one says that they are **cohomologous** to each other if there exists a measurable function $u: \mathbf{T} \to \mathbf{G}$ such that

$$f_1^{(n)}(t) = f_2^{(2)}(t) + u(t + k\alpha) - u(t).$$

For any cocycle $f(t)$, let $\mathbf{G}_f$ denote the closed subgroup of $\mathbf{G}$ generated by

$$\{f^{(n)}(t) \colon t \in \mathbf{T} \text{ and } n \in \mathbf{Z}\}.$$

A cocycle $f(t)$ is called **minimal** if there is no cocycle $f_0(t)$ cohomologous to f such that $\mathbf{G}_{f_0} \subset \mathbf{G}_f$ but $\mathbf{G}_{f_0} \neq \mathbf{G}_f$.

For any cocycle $f(t)$, one can define a measure preserving transformation S (sometimes we use S_f for emphasizing the cocycle) on $\mathbf{T} \times \mathbf{G}$ by

$$S(t, \mathbf{x}) = (t + \alpha, \mathbf{x} + f(t)).$$

Now we use a known result to end this section. A proof of the following theorem can be found in [11, page 391].

THEOREM 2.2: *With the above definitions,*

(i) *any cocycle is cohomologous to a minimal cocycle;*

(ii) *a cocycle $f(t)$ is minimal and $\mathbf{G}_f = \mathbf{G}$ if and only if S_f is ergodic on $\mathbf{T}\times\mathbf{G}$;*

(iii) *if $f_1(t)$ and $f_2(t)$ are cohomologous, then $\mathbf{G}_{f_1} = \mathbf{G}_{f_2}$.*

3. Ergodicity of skew products

In this section, we will use results on continued fractions for irrational numbers and uniform distributions of sequences in $\mathbb{R}^m$. More detail can be found in [4, 5].

For any $x \in \mathbb{R}^m$, $[\mathbf{x}] = ([x_1], [x_2], \ldots, [x_m])$ and $]\mathbf{x}[= \mathbf{x} - [\mathbf{x}]$. For any $x \in \mathbf{R}$, we will use $|\, x|$ to denote the distance from x to the nearest integer. The following result can be found in [7, page 161].

THEOREM 3.1: *Let $\{A_n\}$ be a sequence of nonsingular $m \times m$ matrices with integer entries and, for fixed n and $k = 1, 2, \ldots, n$, let $h_k^{(n)}$ be the number of integers j $(1 \le j \le n)$ such that $\det(A_j - A_k) = 0$. If there are two positive constants ε and c such that*

$$\max_{1\le k\le n} h_k^{(n)} = h^{(n)} \le \frac{cn}{(\log n)^{1+\varepsilon}},$$

then $\{]A_n\mathbf{v}[\}$ is uniformly distributed in $[0,1)^m$ for a.e. $\mathbf{v} \in [0,1)^m$.

COROLLARY 3.2: *Let $\{p_n/q_n\}$ be the partial convergents of an irrational number $\alpha \in [0,1)$. Then for a.e. $\mathbf{v} \in [0,1)^m$, $\{]q_n\mathbf{v}[\}$ is uniformly distributed in $[0,1)^m$.*

Proof: Let I be the $m \times m$ identity matrix and let $A_n = q_n I$. Then $q_n\mathbf{v} = A_n\mathbf{v}$. It is clear that $h_k^{(n)} = 1$. So the conditions in Theorem 3.1 are satisfied. Therefore the corollary follows. ∎

In [3, pages 8–9], Katznelson proved the following proposition, though he did not summarize the results in a theorem.

PROPOSITION 3.3: *Let $0 < \alpha < \frac{1}{2}$ be an irrational number with continued fraction convergents $\{p_n/q_n\}$ and let $\mathcal{P}_n(\alpha)$ be the set of right half-open partition intervals of $\mathbf{T}$ defined by $\{j\alpha\colon j = 0, 1, \ldots, q_n - 1\}$. Then at most $q_{n-1}+1$ intervals in $\mathcal{P}_n(\alpha)$ have length $|\, q_{n-1}\alpha| \,+ |\, q_n\alpha|$ and the rest of the intervals in $\mathcal{P}_n(\alpha)$ have length $|\, q_{n-1}\alpha|$.*

For any interval J in the partition $\mathcal{P}_n(\alpha)$ defined in Proposition 3.3, it is clear that $|J| < 1/q_n$. Since

$$2|\, q_{n-1}\alpha| \geq |\, q_n\alpha| + |\, q_{n-1}\alpha| \geq |J|,$$

we have $2q_n|\, q_{n-1}\alpha| \geq 1$. Therefore the following corollary follows.

COROLLARY 3.4: *For any $J \in \mathcal{P}_n(\alpha)$, $1/(2q_n) \leq |J| \leq 1/q_n$.*

Let $\mathbf{G} = \{g_0, g_1, g_2, \ldots, g_m\}$ be a finite abelian group with identity $e = g_0$ and let $\mathbf{T}$ be an one-dimensional torus (which is also sometimes denoted by $[0,1)$). Let $\alpha \in [0,1)$ be an irrational number. For any $0 < t_1 < t_2 < \cdots < t_m < 1$, define

$$f_{t_1,t_2,\ldots,t_m}(t) = \begin{cases} e & \text{if } 0 \leq t < t_1 \text{ or } t_m \leq t < 1, \\ g_i & \text{if } t_{i-1} \leq t < t_i \text{ for } i = 2, 3, \ldots, m. \end{cases}$$

For every $(t_1, t_2, \ldots, t_m) \in [0,1)^m$, one can define a measure preserving transformation $S_{t_1,t_2,\ldots,t_m}$ on $\mathbf{T} \times \mathbf{G}$ by:

$$S_{t_1,t_2,\ldots,t_m}(t,g) = (t + \alpha, f_{t_1,t_2,\ldots,t_m}(t) + g).$$

PROPOSITION 3.5: *Let $0 < \alpha < 1/2$ be an irrational number with continued fraction $[a_1, a_2, \ldots]$ and convergents $\{p_n/q_n\}$. Assume that there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} = \infty$. Then for any $0 < t_1 < t_2 < \cdots < t_m < 1$ satisfying that $\{]q_{n_k}(t_1, t_2, \ldots, t_m)[\}$ is uniformly distributed in $[0,1)^m$, $S_{t_1,t_2,\ldots,t_m}$ is ergodic.*

Proof: We here only prove the proposition for the case $\lim_{n\to\infty} a_n = \infty$. The more general situation can be handled in the same way.

For convenience, we use S and $f(t)$ to denote $S_{t_1,t_2,\ldots,t_m}$ and $f_{t_1,t_2,\ldots,t_m}(t)$, respectively. Since $\{]q_n(t_1, t_2, \ldots, t_m)[\}$ is uniformly distributed in $[0,1)^m$, there exists a subsequence $\{q_{n_k}\}$ so that

$$]q_{n_k}(t_1, t_2, \ldots, t_m)[\to \left(\frac{1}{8m}, \frac{2}{8m}, \ldots, \frac{m}{8m}\right).$$

We claim that for any sufficiently large k and for any $i = 1, 2, \ldots, m$,

$$(1) \qquad \mu(\{t\colon f^{(q_{n_k})}(t) = g_i\}) \geq \frac{1}{16m}.$$

Let $\mathcal{P}_n(\alpha)$ be the set of right half-open partition intervals of $\mathbf{T}$ defined by

$$\{]j\alpha[\colon j = 0, 1, \ldots, q_n - 1\}.$$

Let $A_i = \{]t_i + j\alpha[: j = 0, 1, \ldots, m\}$ for $i = 1, 2, \ldots, m$. For any $J \in \mathcal{P}_n(\alpha)$, $|J|$ will be either $\| q_{n-1}\alpha\|$ or $\| q_{n-1}\alpha\| + \| q_n\alpha\|$. Then $|J \cap A_i| \leq 2$ for $i = 1, 2, \ldots, m$. By Proposition 3.3, there is $\mathcal{Q}_n \subset \mathcal{P}_n$ with $|\mathcal{Q}_n| \geq q_n - mq_{n-1}$ such that, for any $J \in \mathcal{Q}_n$, $|J \cap A_i| = 1$ for $i = 1, 2, \ldots, m$.

For any n_k and any $J = [a, b) \in \mathcal{Q}_{n_k}$, let $a_i = J \cap A_i$. Since

$$\alpha = \frac{p_{n_k}}{q_{n_k}} + \frac{r_{n_k}}{a_{n_k} q_{n_k}^2},$$

there is a j' with $-q_{n_k} \leq j' \leq q_{n_k}$ such that

$$a_i - a = t_i + j'\alpha = \frac{j'p_{n_k} + [q_{n_k}t_i]}{q_{n_k}} + \frac{]q_{n_k}t_i[}{q_{n_k}} + \frac{j'r_{n_k}}{a_{n_k}q_{n_k}^2} \pmod 1.$$

Since $|a_i - a| \leq 1/q_{n_k}$, $](j'p_{n_k} + [q_{n_k}t_i])/q_{n_k}[$ will be either 0 or $\pm 1/q_{n_k}$. Noticing that $a_i > a$, $]q_{n_k}t_i[\to i/8m$ and $a_{n_k} \to \infty$, we know that when n_k is sufficiently large

$$\frac{j'p_{n_k} + [q_{n_k}t_i]}{q_{n_k}} = 0 \pmod 1 .$$

For some $\varepsilon > 0$, choose k sufficient large so that $1/a_{n_k} < \varepsilon$ and

$$|]q_{n_k}t_i[- \frac{i}{8m}| \leq \varepsilon.$$

Then for $1 \leq i \leq m$,

$$\frac{i}{8m} - 2m\varepsilon < q_{n_k}(a_i - a) < \frac{i}{8m} + 2m\varepsilon.$$

For $0 \leq i \leq m - 1$, let

$$h_i(t) = \begin{cases} e & \text{if } 0 \leq t < t_i \text{ or } t_m \leq t < 1, \\ g_{i+1} - g_i & \text{if } t_i \leq t < t_m. \end{cases}$$

Then $f(t) = \sum_{i=1}^{m-1} h_i(t)$. Assume $a_i = t_i + j_i\alpha$ for $1 \leq i \leq m - 1$. It is clear that for $t \in (a, b) = J$,

$$f^{(q_{n_k})}(t) = f_0 + \sum_{i=1}^{m-1} h_i(t + j_i\alpha),$$

where $f_0 = f^{(q_{n_k})}(a)$. Noticing that

$$\sum_{i=1}^{m-1} h_i(t + j_i\alpha) = \begin{cases} e & \text{if } 0 \leq t < a_1 \text{ or } a_m \leq t < 1, \\ g_i & \text{if } a_{i-1} \leq t < a_i \text{ for } i = 2, 3, \ldots, m, \end{cases}$$

we have that for $i = 1, 2, \ldots, m$,

$$\mu\{t \in J: f^{(q_{n_k})}(t) = g_i\} \geq \frac{1}{q_{n_k}} \left(\frac{1}{8m} - 2m\varepsilon \right).$$

Since $|Q_{n_k}| \geq q_{n_k} - mq_{n_k-1}$ and $q_{n_k-1}/q_{n_k} \sim 1/a_{n_k} \to 0$, (1) is true for $1 \leq i \leq m$.

Now suppose that S is not ergodic. Then there is a nontrivial subgroup $\mathbf{G}_0 \subset \mathbf{G}$ and a function U: $[0,1) \to \mathbf{G}$ such that $f^{(n)}(t) + U(t + n\alpha) - U(t) \in \mathbf{G}_0$ for all n and a.e. $t \in \mathbf{T}$. Therefore, for sufficiently large k,

$$\mu\{t \in [0,1): f^{(q_{n_k})}(t) \notin \mathbf{G}_0\} < \frac{1}{32m}.$$

This contradicts our claim (1). Therefore the proposition follows. ∎

THEOREM 3.6: *Let $[a_1, a_2, \ldots]$ be the continued fraction of $0 < \alpha < 1/2$. If*

$$\sup\{a_i: i = 1, 2, \ldots\} = +\infty,$$

then for a.e. $(t_1, t_2, \ldots, t_m) \in [0,1)^m$ with $0 < t_1 < t_2 < \cdots < t_m$, $S_{t_1, t_2, \ldots, t_m}$ is ergodic.

Proof: We here only prove the theorem for the case $\lim_{n\to\infty} a_n = \infty$. The more general situation can be proved in the same way.

Let p_n/q_n be n-th order convergents of α. By Corollary 3.2, there is a set $I \subset [0,1)^m \backslash \mathbb{Q}^m \alpha$ with full measure such that, for any $(t_1, t_2, \ldots, t_m) \in I$, the sequence $\{]q_n(t_1, t_2, \ldots, t_m)[\}$ is uniformly distributed in $[0,1)^m$. Now the theorem follows from Proposition 3.5. ∎

4. Examples for $K_2(S) \neq K_2(T)$

Let ρ_α be an irrational rotation on $\mathbf{T}$ defined by the irrational number α. For any integer $m > 0$ and any irrational numbers $t_1, t_2, \beta \in [0,1)$ with $t_1 < t_2$, we define a function f: $\mathbf{T} \to \mathbb{Z}_2 = \{0,1\}$ by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < t_1 \text{ or } t_2 \leq t < 1, \\ 1 & \text{if } t_1 \leq t < t_2, \end{cases} \tag{2}$$

and a measure preserving map S on $\mathbf{T} \times \mathbb{Z}_2^m$ by:

$$S(t, x_1, x_2, \ldots, x_m) = (t + \alpha, x_1 + f(t), x_2 + f(t + \beta), \ldots, x_m + f(t + (m-1)\beta)).$$

PROPOSITION 4.1: *Let $[a_1, a_2, \ldots]$ be the continued fraction of $0 < \alpha < 1/2$ such that*

$$\lim_{n\to\infty} a_n = \infty$$

and let $(t_1, t_2, \beta) \in [0,1)^3 \backslash \mathbb{Q}^3$ with $t_1 < t_2$. If $\{]q_{n_k}(t_1, t_2, \beta)[\}$ is uniformly distributed in $[0,1)^3$, then S is ergodic on $\mathbf{T} \times \mathbb{Z}_2^m$ for all $m > 0$.

Proof: Let $F(t) = (f(t), f(t+\beta), \ldots, f(t+(m-1)\beta))$. Assume that S is not ergodic. Then there is a nontrivial subgroup $\mathbf{G} \subset \mathbb{Z}_2^m$ and a function $U: [0,1) \to \mathbb{Z}^m$ such that

$$F^{(n)}(t) + U(t+n\alpha) - U(t) \in \mathbf{G} \tag{3}$$

for all n and a.e. $t \in \mathbf{T}$. Now we will use the similar method used in Proposition 3.5 to get a contradiction.

For $1 \le i' \le m$, let

$$F_{i'1}(t) = \begin{cases} (0,0,\ldots,0) & \text{if } 0 \le t <]t_1 + (i'-1)\beta[, \\ (\underbrace{0,\ldots,0}_{i'-1},1,0,\ldots,0) & \text{if }]t_1+(i'-1)\beta[\le t < 1, \end{cases}$$

and let

$$F_{i'2}(t) = \begin{cases} (0,0,\ldots,0) & \text{if } 0 \le t <]t_2 + (i'-1)\beta[, \\ (\underbrace{0,\ldots,0}_{i'-1},1,0,\ldots,0) & \text{if }]t_2+(i'-1)\beta[\le t < 1. \end{cases}$$

Let $\mathcal{P}_n(\alpha)$ be the set of right half-open partition intervals of $\mathbf{T}$ defined by

$$\{]j\alpha[: j = 0,1,\ldots,q_n - 1\}.$$

Let

$$A_{ii'} = \{]t_i + (i'-1)\beta + j\alpha[: j = 0,1,\ldots,n\}$$

for $i = 1,2$ and $i' = 1,2,\ldots,m$. For any $J \in \mathcal{P}_n(\alpha)$, $|J|$ will be either $|\ q_{n-1}\alpha|$ or $|\ q_{n-1}\alpha|\ +|\ q_n\alpha|$. Then $|J \cap A_{ii'}| \le 2$ for $i = 1,2$ and $i' = 1,2,\ldots,m$. By Proposition 3.3, there is $\mathcal{Q}_n^0 \subset \mathcal{P}_n$ with $|\mathcal{Q}_n| \ge q_n - 2mq_{n-1}$ such that, for any $J \in \mathcal{Q}_n^0$, $|J \cap A_{ii'}| = 1$ for $i = 1,2$ and $i' = 1,2,\ldots,m$. Now we choose $\{n_k\}$ such that

$$]q_{n_k}(t_1,t_2,\beta)[\to \left(\frac{1}{8m}, \frac{1}{4m}, \frac{1}{4m}\right).$$

Since $U(t+q_{n_k}\alpha) - U(t) \to 0$, when k is sufficiently large

$$\mu\{t: U(t+q_{n_k}\alpha) - U(t) \ne 0\} < \frac{1}{128m}.$$

Let

$$\mathcal{Q}^1_{n_k} = \{J \in \mathcal{P}_{n_k} : \mu(\{t: U(t+q_{n_k}\alpha) - U(t) \neq 0\} \cap J) > |J|/64m\}.$$

Then it is clear that for sufficiently large k, $|\mathcal{Q}^1_{n_k}| < q_{n_k}/2$. Now let $\mathcal{Q}_{n_k} = \mathcal{Q}^0_{n_k} \cap \mathcal{Q}^1_{n_k}$. Then

$$|\mathcal{Q}_{n_k}| > \frac{q_{n_k}}{2} - 2mq_{n_k-1}.$$

For a sufficiently large n_k, let $J = [a,b) \in Q_{n_k}$ and let $a_{ii'} = J \cap A_{ii'}$ for $i = 1,2$ and $i' = 1,2,\ldots,m$. Since

$$\alpha = \frac{p_{n_k}}{q_{n_k}} + \frac{r_{n_k}}{a_{n_k} q^2_{n_k}}$$

and $1/(2q_{n_k}) \leq |J| \leq 1/q_{n_k}$ (see Corollary 3.4), there exists $-q_{n_k} \leq j \leq q_{n_k}$ such that

$$a_{ii'} - a = \left] \frac{jp_{n_k} + [t_i q_{n_k}] + [(i'-1)\beta q_{n_k}]}{q_{n_k}} + \frac{]q_{n_k} t_i[}{q_{n_k}} + \frac{](i'-1)\beta q_{n_k}[}{q_{n_k}} + \frac{j' r_{n_k}}{a_{n_k} q^2_{n_k}} \right[.$$

Since $|a_i - a| \leq 1/q_{n_k}$,

$$\frac{jp_{n_k} + [t_i q_{n_k}] + [(i'-1)\beta q_{n_k}]}{q_{n_k}}$$

will be either 0, $\pm 1/q_{n_k}$ or $\pm 2/q_{n_k}$. Noticing that

$$\{]q_{n_k} t_1[\} \to \frac{i}{8m}, \quad \{]q_{n_k} t_2[\} \to \frac{i}{4m}, \quad \text{and} \quad a_{n_k} \to \infty,$$

we have that, when k is sufficiently large,

$$](i'-1)\beta q_{n_k}[= (i'-1)]\beta q_{n_k}[$$

and

$$\left] \frac{jp_{n_k} + [t_i q_{n_k}] + [(i'-1)\beta q_{n_k}]}{q_{n_k}} \right[= 0.$$

Therefore when k is sufficiently large,

$$\frac{8i'-5}{32m} < q_{n_k}(a_{1i'} - a) < \frac{8i'-3}{32m}$$

and

$$\frac{4i'-1}{32m} < q_{n_k}(a_{2i'} - a) < \frac{4i'+1}{32m}.$$

This gives us a partition

$$a < a_{11} < a_{21} < a_{12} < a_{22} < \cdots < a_{1m} < a_{2m} < \frac{a+b}{2}$$

for interval J such that the length of each interval of the partition of J is longer than $1/32mq_{n_k}$. Since

$$F^{(n_k)}(t) = F^{(n_k)}(a) + \sum_{i'=1}^{m}(F_{i'1}(t + a_{1i'}) + F_{i'1}(t + a_{2i'})),$$

we nave that, for $t \in (a_{1i'}, a_{2i'})$,

$$F^{(n_k)}(t) = F^{(n_k)}(a) + (\underbrace{0, \ldots, 0}_{i'}, 1, 0, \ldots, 0)$$

and for $t \in (a_{2i'}, a_{1(i'+1)})$, $F^{(n_k)}(t) = F^{(n_k)}(a)$. Noticing that $J \in \mathcal{Q}_{n_k}$ and the length of each interval of the partition of J, one has

$$\langle F^{(n_k)}(t) \colon t \in J \backslash \{t \colon U(t + q_{n_k}\alpha) - U(t) \neq 0\}\rangle = \mathbb{Z}_2^m.$$

This contradicts (3). ∎

By Corollary 3.2, we have an immediate corollary.

COROLLARY 4.2: *Let $[a_1, a_2, \ldots]$ be the continued fraction for $0 < \alpha < 1/2$ with $\lim_{n\to\infty} a_n = \infty$. Then, for a.e. $(t_1, t_2, \beta) \in [0, 1)^3$ with $t_1 < t_2$, S is ergodic.*

COROLLARY 4.3: *Let $[a_1, a_2, \ldots]$ be the continued fraction for $0 < \alpha < 1/2$ with $\lim_{n\to\infty} a_n = \infty$ and let $(t_1, t_2, \beta) \in [0, 1)^3$ with $t_1 < t_2$. If $\{]q_n(t_1, t_2, \beta)[\}$ is uniformly distributed in $[0, 1)^3$, then S on $\mathbf{T} \times \mathbb{Z}_2^\infty$ defined by*

$$T(t, \ldots, x_{-1}, x_0, x_1, \ldots) = \\ (t + \alpha, \ldots, x_{-1} + f(t - \beta), x_0 + f(t), x_1 + f(t + \beta), \ldots)$$

is ergodic.

Proof: For any $j \in \mathbb{Z}$, let $\gamma_j(\ldots, x_{-1}, x_0, x_1, \ldots) = 1 - 2x_j$. Then for any $j_1, j_2, \ldots, j_m$

$$\gamma_{(j_1 j_2 \cdots j_m)} = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_m}$$

give all characters on $\mathbb{Z}_2^\infty$.

Assume S is not ergodic. There exists a nonconstant S-invariant function $h \in L^2(\mathbf{T} \times \mathbb{Z}_2^\infty)$. Assume that

$$h(t, \ldots, x_{-1}, x_0, x_1, \ldots) = \sum_{j_1, j_2, \cdots, j_m} a_{j_1, j_2, \ldots, j_m}(t) \gamma_{(j_1 j_2 \cdots j_m)}(\ldots, x_{-1}, x_0, x_1, \ldots).$$

Since h is nonconstant, we assume $a_{j_1, j_2, \ldots, j_m}(t) \neq 0$ for some $j_1, j_2, \ldots, j_m$. Let $N \in \mathbb{Z}$ such that $|j_i| \leq N$ for $i = 1, 2, \ldots, m$ and let S_N be the measure preserving transformation on $\mathbf{T} \times \mathbb{Z}_2^{2N+1}$ defined by

$$S_N(t, x_{-N}, \ldots, x_{-1}, x_0, x_1, \ldots, x_N) = (t + \alpha; x_{-N} + f(t - N\beta), \ldots, x_0 + f(t), x_1 + f(t + \beta), \ldots, x_N + f(t + N\beta)).$$

Then

$$h_N(t; x_{-N}, \ldots, x_{-1}, x_0, x_1, \ldots, x_N) = a_{j_1, j_2, \ldots, j_m}(t)(1 - 2x_{j_1}) \cdots (1 - 2x_{j_m})$$

is a nonconstant S_N-invariant function. This contradicts Proposition 4.1. ∎

Let $X = \mathbf{T} \times \mathbb{Z}_2^\infty$ and let $[a_1, a_2, \ldots]$ be the continued fraction of $0 < \alpha < 1/2$ with $\lim_{n \to \infty} a_n = \infty$. Let p_n/q_n be the convergents of α. Choose $(t_1, t_2, \beta) \in [0, 1)^3$ such that $t_1 < t_2$ and $\{]q_n(t_1, t_2, \beta)[\}$ is uniformly distributed in $[0, 1)^3$. Define $f(t)$ as (2) and

$$\mathbf{F}(t) = (\ldots, f(t - \beta), f(t), f(t + \beta), \ldots).$$

Let τ be the shift on $\mathbb{Z}_2^\infty$, i.e.

$$(\tau \mathbf{x})_i = x_{i+1} \qquad \text{for } \mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathbb{Z}_2^\infty.$$

Now we can define two measure-preserving transformations S and T on X by

$$S(t, \mathbf{x}) = (t + \alpha, \mathbf{x} + \mathbf{F}(t)) \quad \text{and} \quad T(t, \mathbf{x}) = (t + \beta, \tau \mathbf{x}).$$

By Corollary 4.3, S is ergodic. Since τ is mixing, T is also ergodic. Noticing that $\mathbf{F}(t + \beta) = \tau \mathbf{F}(t)$, we have

$$\begin{aligned} ST(t, \mathbf{x}) &= (t + \alpha + \beta, \mathbf{F}(t + \beta) + \tau \mathbf{x}) \\ &= (t + \alpha + \beta, \tau(\mathbf{F}(t) + \mathbf{x})) = TS(t, \mathbf{x}). \end{aligned}$$

Therefore S and T are commuting ergodic measure preserving transformations on X. By Examples 1 and 2 in §2, we know that $K_2(S) = X$ and $K_2(T) = K_1(T) = \mathbf{T}$. Therefore:

THEOREM 4.4: *There exist ergodic commuting measure preserving transformations S and T on a probability space X such that $K_2(S) \neq K_2(T)$.*

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